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A.E. BROUWER

# OPTIMAL PACKINGS OF $K_4$'s INTO A $K_n$

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# Optimal packings of $K_4$'s into a $K_n$ *)

by

A.E. Brouwer

ABSTRACT

In this paper we determine the maximum cardinality of a packing of $K_4$'s into $K_n$, that is, construct optimal constant weight codes with weight 4 and minimum distance 6.

KEYWORDS & PHRASES: *packing, group-divisible design.*

*) This report will be submitted for publication elsewhere

## 0. INTRODUCTION

Let $I_n$ be a finite set of n elements. For $v \geq k \geq t$ let $D(t,k,v)$ be the largest integer b such that there exist b subsets $B_1,\ldots,B_b$ of $I_n$, each of k elements, such that every t-element subset of $I_n$ is contained in at most one of them. Our object is to determine $D(2,4,v)$.
Define

$$J(2,4,v) = \begin{cases} \left[\frac{v}{4}\left[\frac{v-1}{3}\right]\right] - 1 & \text{for } v \equiv 7 \text{ or } 10 \pmod{12} \\ \left[\frac{v}{4}\left[\frac{v-1}{3}\right]\right] & \text{otherwise.} \end{cases}$$

<u>Theorem.</u>

(i) $D(2,4,v) = J(2,4,v)$
*iff* $v \notin \{8,9,10,11,17,19\}$

(ii) $D(2,4,v) = J(2,4,v) - 1$
*iff* $v \in \{9,10,17\}$

(iii) $D(2,4,v) = J(2,4,v) - 2$
*iff* $v \in \{8,11,19\}$.

Here $D(2,4,18) = 22$ follows from packings constructed by S. LIN and H.R. PHINNEY [17], $D(2,4,19) = 25$ follows from the work of H.R. PHINNEY and D. STINSON [17], $D(2,4,17) = 20$ was proved in A.E. BROUWER [4], while the values of $D(2,4,v)$ for $v \in \{8,9,10,11\}$ are easily determined by hand. Below we shall prove

$$D(2,4,v) = J(2,4,v) \quad \textit{for all} \quad v \notin \{8,9,10,11,17,18,19\}.$$

Independently partial results on this problem have been obtained by J.-C. BERMOND & J. NOVÁK [2] and by R.C. MULLIN [17]. Equality of $D(2,4,v)$ and $J(2,4,v)$ for sufficiently large v was shown by R.M. WILSON [18].

Concerning the terminology, we shall sometimes use graph-thoeretic concepts, identifying $I_n$ with the vertices of the complete graph $K_n$, and the unordered pairs $\subset I_n$ with its edges. Thus an r-factor is a regular subgraph of $K_n$ with valency r and a $\Delta$-factor is a collection of triples partitioning $I_n$ (sometimes identified with the 2-factor covered by it).

## Optimal packings.

Theorem 1. *Let* $v \notin \{8,9,10,11,17,18,19\}$. *Then we have*

$$D(2,4,v) = J(2,4,v) = \begin{cases} \frac{1}{12}\, v(v-1) & \text{if } v \equiv 1 \text{ or } 4 \pmod{12} \\ \frac{1}{12}\, v(v-2) & \text{if } v \equiv 2 \text{ or } 8 \pmod{12} \\ \frac{1}{12}\, v(v-3) & \text{if } v \equiv 0 \text{ or } 3 \pmod{12} \\ \frac{1}{12}\, (v(v-1) - 18) & \text{if } v \equiv 7 \text{ or } 10 \pmod{12} \\ \frac{1}{12}\, (v(v-2) - 3) & \text{if } v \equiv 5 \text{ or } 11 \pmod{12} \\ \frac{1}{12}\, (v(v-3) - 6) & \text{if } v \equiv 6 \text{ or } 9 \pmod{12} \end{cases}$$

PROOF.

(i) $D(2,4,v) \le J(2,4,v)$ follows from an easy counting argument. (This is a special case of the Johnson bound, see e.g. JOHNSON [10] or SCHONHEIM [16]).

(ii) Let a dense packing be a packing with $J(2,4,v)$ 4-sets. Then by (i) a dense packing is optimal (if it exists). That dense packings indeed exist (for $v \notin \{8,9,10,11,17,18,19\}$) is shown by constructing certain designs, using the recursive techniques of HANANI and WILSON. We consider 6 cases according to the residue of $v \pmod{12}$.

a) $v \equiv 1$ or $4 \pmod{12}$.
In this case a dense packing is a Steiner system $S(2,4,v)$, that is, a balanced incomplete block design with $k = 4$ and $\lambda = 1$. These designs have been constructed by HANANI [6].

b) $v \equiv 2$ or $8 \pmod{12}$.
In this case a dense packing covers all pairs of $I_v$ except for $\frac{1}{2}v$ disjoint pairs. In other words, the 4-sets of the packing from the blocks of a group-divisible design $GD(4,1,2;v)$. [We follow the notation of HANANI [7]; a $GD(K,\lambda,M;v)$ is a pairwise balanced design $B(K \cup M,\lambda;v)$ with a distinguished parallel class of blocks (called groups) with sizes in $M$, while all other blocks (called the blocks of the group-divisible design) have sizes in $K$. Instead of $GD(\{k\},\lambda,\{m\};v)$ we write $GD(k,\lambda,m;v)$. If an element of $K$ or $M$ is starred this means that

there is exactly one block or group of this size. $B(K,\lambda)$ denotes the set of all v for which a $B(K,\lambda;v)$ exists; the same notation is used for other types of design.] But in BROUWER, HANANI & SCHRIJVER [5] the set $GD(4,\lambda,m)$ is determined for all m and $\lambda$. In particular they prove:

Theorem 2. $GD(4,1,2) = \{v \mid v \equiv 2 \pmod 6\} \setminus \{8\}$.

c) $v \equiv 0$ or $3 \pmod{12}$.

In this case a dense packing covers all pairs of $I_v$ except for v pairs that form a 2-factor (i.e. cover each point twice). Such a packing may be obtained from a $S(2,4,v+1)$ by deleting one point and all the blocks containing it. [Now the 2-factor is a collection of $\frac{1}{3}$ v triangles.]

d) $v \equiv 7$ or $10 \pmod{12}$.

In this case a dense packing covers all pairs of $I_v$ (edges of $K_v$) except for 9 edges, where these nine edges form a regular graph of valency 3 on 6 vertices. Such a packing may be obtained from a $B(\{4,7^*\},1;v)$ by replacing the block $\{x_0,\ldots,x_6\}$ of size 7 of such a design by the two four-tuples $\{x_0,x_1,x_2,x_3\}$ and $\{x_0,x_4,x_5,x_6\}$. [The nine uncovered edges are the elements of $\{x_1,x_2,x_3\}\times\{x_4,x_5,x_6\}$ and form a $K_{3,3}$.] Note that indeed

$\frac{1}{6}(\binom{v}{2}-\binom{7}{2})+2 = \frac{1}{12}(v(v-1)-18) = J(2,4,v)$. The existence of the required design is assured by

Theorem 3. *A* $B(\{4,7^*\},1;v)$, *that is, a pairwise balanced design on* v *points with blocks of size* 4 *and exactly one block of size* 7 (*and* $\lambda = 1$), *exists iff* $v \equiv 7$ *or* $10 \pmod{12}$, $v \neq 10,19$.

This theorem is proved below (section 4).

e) $v \equiv 5$ or $11 \pmod{12}$

In this case a dense packing covers all pairs of $I_v$ except for $\frac{v+3}{2}$ edges, where these $\frac{v+3}{2}$ edges form $\frac{v-5}{2}$ disjoint pairs and a star on 5 vertices (a $K_{1,4}$). Such a packing may be obtained from a $GD(4,1,\{2,5^*\};v)$ by taking its blocks and adding the 4-set $\{x_1,x_2,x_3,x_4\}$ if $\{x_0,\ldots,x_4\}$ is the unique group of size five. Note that indeed $\frac{1}{6}(\binom{v}{2}-\binom{5}{2}-\frac{v-5}{2})+1 = \frac{1}{12}(v(v-2)-3) = J(2,4,v)$. The existence of the required design is assured by

Theorem 4. *A* GD(4,1,{2,5*};v), *that is, a group-divisible design on* v *points with blocks of size* 4 *and groups of size* 2 *and exactly one group of size* 5 *(and* $\lambda = 1$*), exists iff* $v \equiv 5 \pmod 6$, $v \neq 11,17$. This theorem is proved below (section 5).

f) $v \equiv 6$ or $9 \pmod{12}$.

In this case a dense packing covers all pairs of $I_v$ except for $v + 3$ pairs (that form a graph on v points with valency $\equiv 2 \pmod 3$ at each vertex). Such a packing may be obtained from a dense packing on $v + 1$ points by removing one point and all the blocks containing it. The point that is removed should be one of the six incident with an uncovered edge (see case d). Note that indeed $J(2,4,v+1) - \frac{v-3}{3} =$ $= \frac{1}{12}(v(v+1)-18) - \frac{v-3}{3} = \frac{1}{12}(v(v-3)-6) = J(2,4,v)$. □

At this point we can already prove the easy halves of theorems 3 and 4:

LEMMA 1.

(i) $B(\{4,7^*\},1) \subset \{v \mid v \equiv 7 \text{ or } 10 \pmod{12}\}\setminus\{10,19\}$

(ii) $GD(4,1,\{2,5^*\}) \subset \{v \mid v \equiv 5 \pmod 6\}\setminus\{11,17\}$.

PROOF.

(i) Consider the blocks of a $B(\{4,7^*\}),1;v)$ containing a fixed point p. p has valency $v - 1$ in $K_v$ and each block covers 3 or 6 of these $v - 1$ edges, hence $v - 1 \equiv 0 \pmod 3$. Next observe that $\binom{4}{2} = 6$ is even and $\binom{7}{2} = 21$ is odd, so that $K_v$ must have an odd number of edges, i.e. $v \equiv 2$ or $3 \pmod 4$. Finally let $v > 7$. Then each block of size 4 covers at least 3 edges disjoint from the block of size 7 and at most 3 edges intersecting it. Hence $\binom{v-7}{2} \geq 7.(v-7)$, i.e. $v \geq 22$.

(ii) If we now consider a $GD(4,1,\{2,5^*\};v)$ it follows by the same arguments that $v - 1 \equiv 1 \pmod 3$ and $v \geq 17$. Since the groups form a partition $v \equiv 1 \pmod 2$. If a $GD(4,1,\{2,5^*\};17)$ existed then by counting it follows that each block intersects the group of size 5 so that removal of this group and its points yields a resolvable group divisible design RGD(3,1,2;12) (also called a Nearly Kirkman Triple System NKTS(12)). But such a design does not exist (KOTZIG & ROSA [11], cf. section 3). □

REMARK. WILSON [20] shows that $B(\{4,7,10,19\},1) = \{v \mid v \equiv 1 \pmod 3\}$, essentially using the availability of blocksize 10 (especially for $v = 31$).

From theorem 3 it follows that $B(\{4,7\},1) = \{v \mid v \equiv 1 \pmod 3)\}\setminus\{10,19\}$, a strengthening of Wilson's result.

## 2. TRANSVERSAL DESIGNS

A transversal design $T(k,\lambda;m)$ is a group-divisible $GD(k,\lambda,m;km)$ (that is, each block is a transversal of the collection of groups) (cf. HANANI [7]). A $T(4,1;n)$ exists iff $n \neq 2,6$ (cf. BOSE, PARKER & SHRIKHANDE [3]); a $T(5,1;n)$ is known for $n \notin \{2,3,6,10,14,42\}$ (for $n > 42$ see HANANI [7] p. 277, for $n \equiv 0$ or $1 \pmod 4$ see MILLS [13], for $n = 15$ see HEDAYAT [9], for $n = 30$ see WILSON [19], solutions for $n \in \{18,22,26,34,38\}$ have been found by S.M.P. WANG [21]; also see VAN LINT [21]). If we take a transversal design $T(5,1;t)$ and remove $t-h$ points of one group (where $0 \le h \le t$) we get a $GD(\{4,5\},1,\{h,t\};4t+h)$ (called a truncated transversal design). Call the underlying pointset of this design $X$ (so that $|X| = 4t+h$) and construct a $GD(4,1,\{3h,3t\};3(4t+h))$ on $X \times I_3$ by taking for each group $G$ of the original design a new group $G \times I_3$, and for each block $B$ the blocks of a $GD(4,1,3;3.|B|)$ constructed on the set $B \times I_3$ in such a way that it has groups $\{b\} \times I_3$ for $b \in B$. [Note that a $GD(4,1,3;12)$ is obtained by removing one point from the projective plane $PG(2,3)$, while a $GD(4,1,3;15)$ is obtained by removing one point from the affine plane $AG(2,4)$.] Using this construction we can give a recursive construction for optimal packings:

<u>LEMMA 2</u>. *If* $\{3h+7,3t+7\} \subset B(\{4,7^*\},1)$ *and* $t \ge h$ *then* $12t+3h+7 \in B(\{4,7^*\},1)$

<u>PROOF</u>. From $3t+7 \in B(\{4,7^*\},1)$ it follows that $t \equiv 0$ or $1 \pmod 4$ (lemma 1) hence a $T(5,1;t)$ exists. If we construct a $GD(4,1,\{3h,3t\};12t+3h)$ on $X \times I_3$ as above, then adding a block $Z$ of size 7 disjoint from $X \times I_3$ and replacing each group $G$ by a $B(\{4,7^*\},1;|G|+7)$ on the set $G \cup Z$ which has $Z$ as its block of size 7 yields a $B(\{4,7^*\},1;12t+3h+7)$. □

And in the same way we see that

<u>LEMMA 3</u>. *If* $\{3h+5,3t+5\} \subset GD(4,1,\{2,5^*\})$ *and* $t \ge h$ *and* $t \in T(5,1)$ *then* $12t+3h+5 \in GD(4,1,\{2,5^*\})$.

But here we can avoid the requirement $3t+5 \in GD(4,1,\{2,5^*\})$ by proceeding somewhat differently:

LEMMA 4. *Let* $h < t$, $t$ *even*, $t \in T(5,1)$ *and* $3h+2 \in GD(4,1,\{2,5^*\})$. *Then* $12t+3h+2 \in GD(4,1,\{2,5^*\})$.

PROOF. Since $t$ is even, $3t+2 \in GD(4,1,2)$. Start with a $GD(4,1,\{3h^*,3t\};12t+3h)$, add a group $Z$ of size 2, and replace each group $G$ of size $3t$ by a $GD(4,1,2;3t+2)$ on the set $G \cup Z$ which has $Z$ as a group; further replace the group $H$ of size $3h$ by a $GD(4,1,\{2,5^*\};3h+2)$ which has $Z$ as a group, except when $h = 1$ in which case we take $H \cup Z$ as a group of size 5, and do not take $Z$ as a group. This yields a $GD(4,1,\{2,5^*\};12t+3h+2)$. □

Yet another way of positioning the group of size 5 is used to get

LEMMA 5. *Let* $s \neq 1$. *Then* $24s+5 \in GD(4,1,\{2,5^*\})$.

PROOF. Let $X = (I_{6s+1} \times I_4) \cup \{\infty\}$. Construct a transversal design $T(4,1;6s+1)$ on the set $X\backslash\{\infty\}$ with groups $I_{6s+1} \times \{i\}$, $i \in I_4$ which has among its blocks $\{a\} \times I_4$ for some $a \in I_{6s+1}$. Replace each group $I_{6s+1} \times \{i\}$ by a group-divisible design $GD(4,1,2;6s+2)$ on the set $I_{6s+1} \times \{i\} \cup \{\infty\}$ which has $\{(a,i),\infty\}$ among its groups. Finally replace the block $\{a\} \times I_4$ and the groups $\{(a,i),\infty\}$ by the group $\{a\} \times I_4 \cup \{\infty\}$. This yields a $GD(4,1,\{2,5^*\};24s+5)$ on $X$. □

A similar construction yields

LEMMA 6. *Let* $v \equiv 7$ or $43 \pmod{48}$. *Then* $v \in B(\{4,7^*\},1)$.

PROOF. Let $v = 4t+3$, then $t+3 \equiv 1$ or $4 \pmod{12}$ and hence $t+3 \in B(4)$. Also $t \neq 2,6$ so that we may construct a transversal design $T(4,1;t)$ on $I_t \times I_4$ which has $\{a\} \times I_4$ among its blocks and $I_t \times \{i\}$ as its groups ($i \in I_4$). Let $X = (I_t \times I_4) \cup I_3$ and construct a $B(\{4,7^*\},1;v)$ on $X$ by replacing each group $I_t \times \{i\}$ by the blocks of a $B(4;t+3)$ on $(I_t \times \{i\}) \cup I_3$ that has $\{(a,i)\} \cup I_3$ among its blocks, and then replacing the five blocks $\{a\} \times I_4$ and $\{(a,i)\} \cup I_3$ ($i \in I_4$) by the single block $(\{a\} \times I_4) \cup I_3$ of size 7. □

## 3. COMPLETION OF RESOLVABLE DESIGNS

A resolvable (transversal, pairwise balanced or group-divisible) design is a design of which the blocks can be partitioned into parallel classes. We write RT, RB or RGD with the appropriate parameters. Resolvable pairwise balanced designs with $k = 3$ and $\lambda = 1$ are called Kirkman triple systems, and RAY-CHAUDHURI & WILSON [15] proved that

$$RB(3,1) = \{v \mid v \equiv 3 \pmod 6\}.$$

HANANI, RAY-CHAUDHURI & WILSON [8] proved the existence of resolvable quadruple systems:

$$RB(4,1) = \{v \mid v \equiv 4 \pmod{12}\}.$$

Resolvable group divisible designs with $k = 3$, $m = 2$ and $\lambda = 1$ are called Nearly Kirkman Triple Systems (KOTZIG & ROSA [11]), and BAKER & WILSON [1] proved $RGD(3,1,2) \supset \{v \mid v \equiv 0 \pmod 6\} \setminus \{6,12,84,102,174\}$.
[Surely $RGD(3,1,2) = \{v \mid v \equiv 0 \pmod 6\} \setminus \{6,12\}$ but solutions for the remaining three cases are not known yet.] If $\mathcal{B}$ is the collection of blocks of some resolvable design on v points and with $\lambda = 1$, and $\mathcal{B} = \bigcup_{j=1}^{r} \mathcal{B}_j$ is a partition into parallel classes, then for $1 \le s \le r$ we can form a design on v+s points by adding new points $\infty_j$ $(1 \le j \le s)$ and replacing each block $B \in \mathcal{B}_j$ by $B \cup \{\infty_j\}$ for $1 \le j \le s$ and adding $\{\infty_j \mid 1 \le j \le s\}$ as a block (or a group in case of a group-divisible design). This process is called partial completion, and completion if $s = r$. In particular, by completing a $RB(3,1;v)$ we find (taking $v = 6t+3$):

LEMMA 7. *Let* $t > 1$. *Then* $9t+4 \in B(\{4,(3t+1)^*\},1)$.

In the same way, by completing a $RGD(3,1,2;v)$ we find (taking $v = 6t$):

LEMMA 8. *Let* $t \notin \{1,2,14,17,29\}$. *Then* $9t-1 \in GD(4,1,\{2,(3t-1)^*\})$.

Finally, (partially) completing a $RB(4,1;v)$ we find (taking $v = 12t+4$):

LEMMA 9. *Let* $1 \le s \le 4t+1$. *Then* $12t+s+4 \in B(\{4,5,s^*\},1)$. *(Here in case* $s \in \{4,5\}$, *the star means that there is one distinguished block of size* s, *and all other blocks have size* 4 *or* 5.)

The first of these lemma's implies that if $3t+1 \in B(\{4,7^*\},1)$ then also $9t+4 \in B(\{4,7^*\},1)$; the second one that if $3t-1 \in GD(4,1,\{2,5^*\})$ then also $9t-1 \in GD(4,1,\{2,5^*\})$ (provided that $t \ne 1,2,14,17,29$). The third one can be used to prove:

LEMMA 10. *Let* $1 \le s \le 4t+1$ *and* $3s+1 \in B(\{4,7^*\},1)$. *Then* $36t+3s+13 \in B(\{4,7^*\},1)$.

PROOF. Given a $B(\{4,5,s^*\},1;v)$ on a set X, construct a $B(\{4,7^*\},1;3v+1)$ on $X \times I_3 \cup \{\infty\}$ by replacing each block B of size 4 or 5 of the original design by the blocks of a $GD(4,1,3;3.|B|)$ on $B \times I_3$ that has groups $\{b\} \times I_3$ $(b \in B)$, replacing the block S of size s by the blocks of a $B(\{4,7^*\},1;3s+1)$ on $(S \times I_3) \cup \{\infty\}$, and adding blocks $(\{a\} \times I_3) \cup \{\infty\}$ for all points $a \in X \backslash S$. □

## 4. PROOF OF THEOREM 3

Let $U = B(\{4,7^*\},1)$, then we have to show that $U = \{v \mid v \equiv 7 \text{ or } 10 \pmod{12}, v \neq 10,19\}$. This will be done by induction on v, i.e. we assume that $w \in U$ for $w < v$, $w \equiv 7$ or $10 \pmod{12}$, $w \neq 10,19$.
First exploit lemma 2 to reduce the problem to a finite one.

Let $v \equiv 7$ or $10 \pmod{12}$. There are 8 cases mod 48:

For $v \equiv 7$ or $10 \pmod{48}$ write $v = 12t+7$. Then $t \equiv 0$ or $1 \pmod 4$ and we may apply lemma 2 (with $h = 0$) to get $v \in U$ unless $3t+7 \in \{10,19\}$, i.e. $v \in \{19,55\}$. $19 \notin U$, and $55 \in U$ follows from lemma 6.

For $v \equiv 22$ or $34 \pmod{48}$ write $v = 12t+3.5+7$. Applying lemma 2 with $h = 5$ yields $v \in U$ unless $v \in \{22,34,70\}$. But $22 \in U$ follows from lemma 7.

For $v \equiv 31$ or $43 \pmod{48}$ write $v = 12t+3.8+7$. Applying lemma 2 with $h = 8$, $t \geq 8$ yields $v \in U$ unless $v \in \{31,43,79,91\}$. But $\{43,91\} \subset U$ by lemma 6.

For $v \equiv 46 \pmod{48}$ write $v = 12t+3.9+7$. Applying lemma 2 with $h = 9$, $t \geq 9$ yields $v \in U$ unless $v \in \{46,94\}$. But $94 \in U$ follows from lemma 7.

For $v \equiv 10 \pmod{48}$ write $v = 12t+3.13+7$. Applying lemma 2 with $h = 13$, $t \geq 13$ yields $v \in U$ unless $v \in \{58,106,154\}$. But $\{106,154\} \subset U$ follows from lemma 10 (with $t = 2$, $s = 7$ and $t = 3$, $s = 11$ respectively).

This reduces the problem to establishing $\{31,34,46,58,70,79\} \subset U$.

In [14] MILLS showed that $70 \in B(\{4,22^*\},1)$ and $79 \in B(\{4,13^*,22^*\},1)$. Since $13 \in B(4,1)$ and $22 \in B(\{4,7^*\},1)$ it immediately follows that $\{70,79\} \subset U$. This leaves four designs to construct; three were made by hand but $31 \in U$ was proved in close cooperation with a PDP 11/45 computer.

A. The case $v = 31$.

Below we produce a $B(\{3,4\},1;24)$ where the blocks of size 3 form 7 parallel classes. Obviously completion of this design yields a $B(\{4,7^*\},1;31)$. Let $X = Z_2 \times Z_2 \times Z_6$ (where $Z_n$ denotes the cyclic group of residues mod n), and take the following blocks:

18 quadruples:

$\{(0,0,0),(0,1,0),(1,0,0),(1,1,0)\}$ mod $(-,-,6)$
$\{(0,0,0),(0,0,3),(1,1,1),(1,1,4)\}$ mod $(2,2,-)$
$\{(0,0,0),(0,0,4),(1,1,5),(0,1,2)\}$ mod $(2,2,-)$
$\{(0,0,1),(0,0,5),(1,1,2),(0,1,3)\}$ mod $(2,2,-)$

7 Δ-factors:

1. $[\{(0,0,0),(0,0,1),(0,0,2)\}, \{(0,0,3),(0,0,4),(0,0,5)\}$ mod $(2,2,-)]$.

2,3. $[\{(0,0,0),(0,0,5),(0,1,1)\}, \{(0,0,2),(1,1,0),(0,1,3)\}$
$\{(1,1,1),(1,1,3),(1,0,4)\}, \{(0,0,4),(1,1,2),(1,0,5)\}$ mod $(-,2,-)$
mod $(2,-,-)$.

4,5. $[\{(0,0,2),(0,0,3),(1,0,4)\}, \{(1,1,2),(1,1,5),(0,1,1)\},$
$\{(0,0,0),(1,0,1),(0,1,4)\}, \{(1,1,0),(1,0,3),(0,1,5)\}$ mod $(-,2,-)]$
mod $(2,-,-)$.

6,7. $[\{(0,0,0),(1,1,3),(0,1,5)\}, \{(0,0,2),(0,0,4),(1,0,0)\},$
$\{(0,0,1),(1,1,5),(1,0,4)\}, \{(0,0,3),(1,1,2),(1,0,1)\}$ mod $(-,2,-)]$
mod $(2,-,-)$

Clearly it is a finite task to check the correctness of this design.

B. The case v = 34.

Let $X = (Z_3 \times Z_9) \cup (I_2 \times Z_3) \cup \{\infty\}$, where the elements of $Z_3 \times Z_9$ are written $(i,j)$ and those of $I_2 \times Z_3$ $[i,j]$.

Take the following blocks:

$\{(i,j),(i+1,j+2),(i+2,j+2),(i+2,j+3)\}$
$\{(i,j),(i+1,j+3),(i+1,j+5),[0,j-i]\}$
$\{(i,j),(i+1,j+4),(i+1,j+8),[1,j]\}$
$\{(i,j),(i,j+3),(i,j+6),\infty\}$ $(j<3)$,

for all $i \in Z_3$, $j \in Z_9$.

C. The case v = 46.

Let $X = (Z_3 \times Z_{13}) \cup (I_2 \times Z_3) \cup \{\infty\}$, and take the following blocks:

$\{(i,j+1),(i,j+3),(i,j+9),(i+1,j)\}$

$\{(i,j+2),(i,j+6),(i,j+5),(i+1,j)\}$

$\{(i,j),(i+1,j+1),(i+2,j+4),[0,i]\}$

$\{(i,j),(i+1,j+2),(i+2,j+7),[1,i]\}$

$\{(0,j),(1,j),(2,j),\infty\}$

for all $i \in Z_3$, $j \in Z_{13}$.

D. The case v = 58.

Let $X = (Z_3 \times Z_{17}) \cup (I_2 \times Z_3) \cup \{\infty\}$, and take the following blocks:

$\{(i,j),(i,j+1),(i,j+4),(i+1,j+5)\}$

$\{(i,j),(i,j+2),(i,j+8),(i+1,j+11)\}$

$\{(i,j),(i,j+5),(i+1,j+2),(i+1,j+12)\}$

$\{(i,j),(i+1,j+8),(i+2,j+7),[0,i]\}$

$\{(i,j),(i+1,j+6),(i+2,j+4),[1,i]\}$

$\{(0,j),(1,j),(2,j),\infty\}$,

for all $i \in Z_3$, $j \in Z_{17}$.

This completes the proof of theorem 3.

## 5. PROOF OF THEOREM 4

Let $V = \{m \mid 6m+5 \in GD(4,1,\{2,5^*\})\}$, then we have to show that $V = \mathbb{N} \setminus \{1,2\}$. This will be done by induction on m, i.e. we assume that $s \in V$ for $s < m$, $s \neq 1,2$. First exploit lemma 4 to reduce the problem to a finite one. We may restate it as

LEMMA 4'. *Let* $h < t$, $h \in V$ *and* $2t \in T(5,1)$, *then* $4t+h \in V$.

Using $2t \in T(5,1)$ if t is even we apply this lemma with the following values of t and h:

| h | t | 4t+h |
|---|---|---|
| 0 | 2+2s | 8+8s |
| 3 | 4+2s | 19+8s |
| 4 | 6+2s | 28+8s |
| 5 | 6+2s | 29+8s |
| 6 | 8+2s | 38+8s |
| 7 | 8+2s | 39+8s |
| 9 | 10+2s | 49+8s |
| 10 | 12+2s | 58+8s |

In particular from $\{0,3,4,5,6,7,9,10\} \subset V$ it follows that $m \in V$ for $m > 50$. We now give various constructions killing the remaining cases.

LEMMA 11. *Let* $m \equiv 0 \pmod 4$. *Then* $m \in V$.

PROOF. For $m \neq 4$ this is just a restatement of lemma 5. We now prove $29 \in GD(4,1,\{2,5^*\})$.

Let $X = (Z_3 \times Z_8) \cup \{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5,\}$.
Take the groups $\{(0,0),(0,4)\} \bmod (3,8)/2$
and $\{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5\}$,
and the blocks

$\{(0,1),\ (0,3),\ (0,4),\ (1,0)\}$
$\{(1,1),\ (1,3),\ (1,4),\ (2,7)\}$
$\{(0,0),\ (2,0),\ (2,2),\ (2,7)\}$
$\{\infty_1,\ (0,0),\ (1,0),\ (2,1)\}$
$\{\infty_2,\ (0,0),\ (1,1),\ (2,6)\}$
$\{\infty_3,\ (0,0),\ (1,2),\ (2,4)\}$
$\{\infty_4,\ (0,0),\ (1,3),\ (2,3)\}$
$\{\infty_5,\ (0,0),\ (1,6),\ (2,5)\}$, all mod $(-,8)$.

[Here $\{(0,0),(0,4)\} \bmod (3,8)/2$ means that adding all elements of $Z_3 \times Z_8$ to the set $\{(0,0),(0,4)\}$ yields the set of groups twice; it is equivalent with $\{(i,j),(i,j+4)\}$, $i \in Z_3$, $j = 0,1,2,3$. We shall need this notation below.] □

LEMMA 12. *Let* $m \equiv 2 \pmod 3$, $m \neq 2$. *Then* $m \in V$.

PROOF. Applying lemma 7 (with $t = 2r$) and using the inductive hypothesis we find that for $r \notin \{1,2,3,7\}$ $3r-1 \in V$. But $\{8,20\} \subset V$ by lemma 11. Presently we prove $5 \in V$ i.e. $35 \in GD(4,1,\{2,5^*\})$.

Let $X = (I_6 \times Z_5) \cup \{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5\}$.

Take the groups $\{(0,0),(1,0)\} \bmod (-,5)$
$\{(2,0),(3,0)\} \bmod (-,5)$
$\{(4,0),(5,0)\} \bmod (-,5)$
and $\{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5\}$.

Take the blocks

$\{(0,0),(0,1),(2,0),(2,2)\}$,
$\{(0,0),(0,2),(3,3),(3,4)\}$,
$\{(1,0),(1,2),(2,1),(2,2)\}$,
$\{(1,0),(1,1),(3,0),(4,0)\}$,
$\{(1,4),(3,0),(3,2),(4,1)\}$,
$\{(0,0),(3,0),(5,0),(5,2)\}$,
$\{(1,0),(2,3),(5,2),(5,3)\}$,
$\{(0,0),(4,3),(4,4),(5,1)\}$,
$\{(2.0),(4,0),(4,2),(5,1)\}$,
$\{\infty_1,(0,0),(2,3),(4,1)\}$,
$\{\infty_1,(1,3),(3,0),(5,3)\}$,
$\{\infty_2,(0,0),(1,1),(4,2)\}$,
$\{\infty_2,(2,0),(3,1),(5,2)\}$,
$\{\infty_3,(0,0),(1,2),(4,0)\}$,
$\{\infty_3,(2,0),(3,4),(5,3)\}$,
$\{\infty_4,(0,0),(1,3),(5,4)\}$,
$\{\infty_4,(2,0),(3,3),(4,1)\}$,
$\{\infty_5,(0,0),(1,4),(5,3)\}$,
$\{\infty_5,(2,0),(3,2),(4,4)\}$,

all mod $(-,5)$.

This yields a design of the required form. □

LEMMA 13. *Let* $m \equiv 5 \pmod 7$. *Then* $m \in V$.

PROOF. We already know $\{5,12,19\} \subset V$. Let $m = 7t+5$ with $t \in V$. Let $X = I_7 \times I_{6t+5}$. Using a $GD(4,1,\{2,5^*\};6t+5)$ on $I_{6t+5}$, take for each of its blocks $B$ the blocks of a $T(4,1;7)$ on $I_7 \times B$ which has groups $I_7 \times \{b\}$ $(b \in B)$; take for each of its groups $G$ of size 2 the blocks and groups of a $GD(4,1,2;14)$ on $I_7 \times G$, and take for the group $H$ of size 5 the blocks and groups of a $GD(4,1,\{2,5^*\};35)$ on $I_7 \times H$. This yields a $GD(4,1,\{2,5^*\};6m+5)$. □

Similarly we have

LEMMA 14. *Let* $m \equiv 0 \pmod 7$, $m \geq 21$. *Then* $m \in V$.

<u>PROOF</u>. Let $m = 7t$ with $t \in V$. Let $X = (I_{3t} \times I_{14}) \cup I_5$. Using a $GD(4,1,2;14)$ on $I_{14}$, take for each of its blocks B the blocks of a $T(4,1;3t)$ on $I_{3t} \times B$ which has groups $I_{3t} \times \{b\}$ $(b \in B)$; take for each of its groups G the blocks and groups of a $GD(4,1,\{2,5^*\};6t+5)$ on $(I_{3t} \times G) \cup I_5$ which has $I_5$ as one of its groups. This yields a $GD(4,1,\{2,5^*\};6m+5)$. □

And in exactly the same way (starting with a $GD(4,1,3;15)$) one proves

<u>LEMMA 15</u>. *Let* $m \equiv 0 \pmod 5$, $m \geq 20$. *Then* $m \in V$.

<u>LEMMA 16</u>. *Let* $m \equiv 2 \pmod 4$, $m \geq 18$. *Then* $m \in V$.

In order to prove this lemma we first need several auxillary designs.

(a) A $GD(4,1,2;20)$ with four pairwise disjoint blocks.
I do not know of any $GD(4,1,2;20)$ with a parallel class, i.e. five pairwise disjoint blocks, but the one constructed in [5] has the four disjoint blocks {00,01,12,14}, {02,04,20,21}, {03,13,34,32}, {10,11,23,33} (where ij is written instead of (i,j)).

(b) A certain class of transversal designs.
We shall need transversal designs $T(4,1;6r+4)$ on $I_{6r+4} \times I_4$ with the following properties:
(α) There is a subset $A \subset I_{6r+4}$ of size 5 such that the blocks contained entirely within $A \times I_4$ form together with at most four blocks of the shape $\{a\} \times I_4$ $(a \in A)$ the blocks of a $T(4,1;5)$ on $A \times I_4$ containing blocks $\{a\} \times I_4$ for all $a \in A$.
(β) There is a point $h \in I_{6r+4} \setminus A$ such that $\{h\} \times I_4$ is a block.
Such designs are constructed in the usual way, starting with a group-divisible design on 6r+4 points and using a $RT(4,1;|B|)$ on $B \times I_4$ for each block B, and a $T(4,1;|G|)$ on $G \times I_4$ for each group G (see e.g. HANANI [7], thm. 3.2). If $r = 2s$ we construct a $GD(\{4,5\},1,\{3,4^*\};12s+4)$ by adding one point "at infinity" to some parallel class of a $RB(4,1;12s+4)$, and then deleting some other point. If $r = 2s+1$ we construct a $GD(\{4,5;7^*\},1,\{3,4\};12s+10)$ by adding 7 points at infinity and then deleting some other point. (This is possible provided $4s+1 \geq 7$, i.e. $s \geq 2$). Now if A is a block of size 5 and H is a group of size 4 intersecting A then for each $h \in H$, $\{h\} \times I_4$ is a block of the transversal design , and the blocks of the transversal design contained

entirely within $A \times I_4$ together with all $\{a\} \times I_4$ $(a \in A)$ form a $T(4,1;5)$ on $A \times I_4$, but since $\{h\} \times I_4$ is contained in $A \times I_4$ for $\{h\} = H \cap A$, we need at most four other blocks $\{a\} \times I_4$. Therefore the required transversal design exists if $r \notin \{0,1,3\}$.

(c) The construction.

Let $X = (I_{6r+4} \times I_4) \cup \{\infty\}$. Take the blocks of a transversal design on $I_{6r+4} \times I_4$ as constructed above, except for $\{h\} \times I_4$ and the blocks contained in $A \times I_4$. Take for $i \in I_4$ the blocks and groups of a $GD(4,1,\{2,5^*\};6r+5)$ on $(I_{6r+5} \times \{i\}) \cup \{\infty\}$ which has $A \times \{i\}$ and $\{(h,i),\infty\}$ among its groups, except for the two groups mentioned.

Next take the blocks and the groups of a $GD(4,1,2;20)$ on $A \times I_4$ constructed in such a way that it contains the at most four blocks $\{a\} \times I_4$ not present in the transversal design, but delete these latter blocks. Finally add $(\{h\} \times I_4) \cup \{\infty\}$ as a group. This yields a $GD(4,1,\{2,5^*\};24r+17)$, hence $4r+2 \in V$ for $r \notin \{0,1,2,3\}$. □

<u>LEMMA 17</u>. $41 \in GD(4,1,\{2,5^*\})$, i.e. $6 \in V$.

<u>PROOF</u>. Let $X = (I_3 \times Z_{12}) \cup \{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5\}$.

Take the groups $\{(i,0),(i,6)\}$ mod$(-,12)/2$ $(i \in I_3)$

and $\{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5\}$.

Take the blocks

$\{(0,0),(0,1),(1,0),(1,2)\}$,
$\{(1,0),(1,1),(2,0),(2,2)\}$,
$\{(0,0),(0,4),(0,7),(1,10)\}$,
$\{(1,0),(1,3),(1,7),(2,10)\}$,
$\{(0,0),(0,2),(2,0),(2,5)\}$,
$\{(0,0),(2,4),(2,7),(2,8)\}$,
$\{\infty_1,(0,0),(1,4),(2,9)\}$,
$\{\infty_2,(0,0),(1,5),(2,1)\}$,
$\{\infty_3,(0,0),(1,7),(2,11)\}$,
$\{\infty_4,(0,0),(1,8),(2,2)\}$,
$\{\infty_5,(0,0),(1,9),(2,6)\}$,

all mod $(-,12)$. □

<u>LEMMA 18</u>. 65 $\in$ GD(4,1,{2,5*}), i.e. 10 $\in$ V.

<u>PROOF</u>. [PDP11] Let $X = Z_3 \times Z_{20} \cup \{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5\}$.

Take the groups {(0,0),(0,10)} mod (3,20)/2

and $\{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5\}$.

Take the blocks {(0,0) ,(0,1),(0,6) ,(0,9)},

{(0,12),(0,8),(1,5) ,(2,0)},

{(0,14),(0,7),(1,10),(2,0)},

{(0,4) ,(0,6),(1,15),(2,0)} all mod (3,20),

and

$\{\infty_1$,(0,0),(1,0) ,(2,18)},

$\{\infty_2$,(0,0),(1,19),(2,19)},

$\{\infty_3$,(0,0),(1,18),(2,0) },

$\{\infty_4$,(0,0),(1,1) ,(2,2) },

$\{\infty_5$,(0,0),(1,2) ,(2,1) }, all mod (-,20). □

Note that this method is generally applicable in the case n ≡ 5 (mod 12): Let $X = (Z_3 \times Z_{4t}) \cup I_5$ and assume that the blocks not intersecting $I_5$ are invariant under $Z_3 \times Z_{4t}$ while the others, though invariant only under $Z_{4t}$, cover a collection of edges which is invariant under $Z_3 \times Z_{4t}$. [In fact, using a similar solution for n = 89 (also found by PDP11), the case n ≡ 5 (mod 12) can be solved completely without recourse to Nearly Kirkman Triple systems.]

<u>LEMMA 19</u>. 23 $\in$ GD(4,1,{2,5*}), i.e. 3 $\in$ V.

<u>PROOF</u>. Let $X = (Z_2 \times Z_3 \times Z_3) \cup \{\infty_0,\infty_1,\infty_2,\infty_3,\infty_4\}$

Take the groups {(0,0,0),(1,0,0)} mod (-,3,3)

and $\{\infty_0,\infty_1,\infty_2,\infty_3,\infty_4\}$,

and the blocks {(0,0,0),(0,1,0),(0,2,0),$\infty_4$} mod (2,-,3)

{(0,0,2),(0,1,1),(0,2,0),$\infty_3$} mod (-,3,-)

{(1,0,0),(1,0,1),(1,0,2),$\infty_3$} mod (-,3,-)

{(0,0,i),(1,0,i+1),(1,1,i+2),$\infty_i$} mod (2,3,-) (i = 0,1,2)

{(0,0,0),(0,0,1),(1,1,1)(1,2,0)} mod (-,3,3). □

<u>LEMMA 20</u>. 47 $\in$ GD(4,1,{2,5*}), i.e. 7 $\in$ V.

PROOF. Let $X = I_6 \times Z_7$ and construct a GD({3,4},1,2;42) on X such that the triples form a 5 Δ-factors (parallel classes). Completion of this design will then yield the required design on 47 points.
Take the groups {(i,0),(i+3,0)} mod (-,7), i = 0,1,2, the Δ-factors
1. {(0,0),(1,5),(5,3)}, {(2,0),(3,2),(4,6)} mod (-,7)
2. {(0,0),(2,4),(4,2)}, {(1,0),(3,5),(5,1)} mod (-,7)
3. {(0,0),(3,4),(5,1)}, {(1,0),(2,1),(4,4)} mod (-,7)
4. {(0,0),(4,1),(5,5)}, {(1,0),(2,2),(3,1)} mod (-,7)
5. {(0,0),(4,4),(5,4)}, {(1,0),(2,3),(3,6)} mdo (-,7)
and the quadruples

{(0,0),(0,1),(1,0),(1,2)}, {(0,0),(0,2),(2,0),(2,1)},
{(0,0),(0,3),(3,1),(3,2)}, {(0,0),(1,3),(1,4),(2,2)},
{(1,0),(1,3),(2,0),(4,1)}, {(2,0),(2,2),(3,0),(5,1)},
{(2,0),(2,3),(4,0),(5,5)}, {(1,0),(3,0),(3,2),(5,0)},
{(2,0),(3,1),(3,4),(4,2)}, {(0,0),(3,3),(4,3),(4,5)},
{(0,0),(4,6),(4,0),(5,2)}, {(1,0),(3,3),(4,6),(4,2)},
{(0,0),(2,3),(5,6),(5,0)}, {(1,0),(3,4),(5,3),(5,6)},
{(1,0),(4,3),(5,2),(5,4)}

all mod (-,7). □

LEMMA 21. $59 \in GD(4,1,\{2,5^*\})$, i.e. $9 \in V$.

PROOF. Let $X = Z_2 \times (Z_3)^3$ and construct a GD({3,4},1,2;54) on X such that the triples form 5 Δ-factors.
Take the groups {(0,0,0,0), (1,0,0,0)} mod (-,3,3,3) and the Δ-factors
1. {(1,0,0,0),(1,2,1,0),(1,1,2,0)} mod (-,3,3,3)/3
{(0,1,2,0),(0,0,0,1),(0,2,1,2)} mod (-,3,3,3)/3
2. {(0,0,0,0),(0,1,1,1),(0,2,2,2)} mod (2,3,3,3)/3
3-5. [{(0,0,0,0),(1,0,1,0),(0,1,2,1)} mod (2,3,-,3)] mod (-,-,3,-)
and the quadruples

{(0,0,0,0),(0,2,1,0),(1,0,0,1),(1,2,1,2)} mod (-,3,3,3)

and

{(0,1,0,0),(1,2,1,0),(0,0,0,2),(0,2,0,2)},
{(0,0,0,0),(1,2,1,0),(0,0,1,2),(0,0,2,2)},
{(0,1,1,0),(1,2,1,0),(0,1,1,2),(0,2,2,2)}, all mod (2,3,3,3). □

LEMMA 22. $83 \in GD(4,1,\{2,5^*\})$, i.e. $13 \in V$.

PROOF. We shall construct a $GD(\{3,4\},1,2;60)$ where the triples form 23 Δ-factors.

(a) Four partitions of $Z_{20}$ each consisting of 5 triples and 5 singletons, such that the triples form the twenty shifts of {0,3,12}, and each point occurs once as a singleton:

1. {0,3,12 },{1,4,13 },{2,5,14 },{6,9,18 },{7,10,19},{8},{11},{15},{16},{17}.
2. {3,6,15 },{4,7,16 },{5,8,17 },{18,1,10},{19,2,11},{0},{ 9},{12},{13},{14}.
3. {8,11,0 },{9,12,1 },{13,16,5},{14,17,6},{15,18,7},{2} { 3},{ 4},{10},{19}.
4. {10,13,2},{11,14,3},{12,15,4},{16,19,8},{17,0,9 },{1} { 5},{ 6},{ 7} {18}.

(b) The construction.
Let $X = I_3 \times Z_{20}$. Take the blocks of a $RT(3,1;20)$ and furthermore on each set $\{i\} \times Z_{20}$ the blocks {0,3,12} and {0,1,5,7} (mod 20) and the groups {0,10} (mod 20)/2. This yields a $GD(\{3,4\},1,2;60)$. We may suppose that one of the parallel classes of the resolvable transversal design was $\{I_3 \times \{j\} \mid j \in Z_{20}\}$, and by (a) we may partition the union of this parallel class and all 'horizontal' triples into 4 parallel classes. Together with the remaining 19 parallel classes of the transversal design this shows that all triples can be partitioned into 23 Δ-factors. □

LEMMA 23. $95 \in GD(4,1,\{2,5^*\})$, i.e. $15 \in V$.

PROOF. Let $X = (I_4 \times I_{23}) \cup I_3$. Since $23 \in B(\{3^*,4,5\},1)$ there exists a transversal design $T(4,1;23)$ on $I_4 \times I_{23}$ that has a subdesign $T(4,1;5)$ on $I_4 \times A$ for some $A \subset I_{23}$ of size 5 (cf. the proof of lemma 16). Take its blocks, except for those in the subdesign. For each $i \in I_4$ take the groups of size 2 and all the blocks of a $GD(4,1,\{2,8^*\};26)$ on $(\{i\} \times I_{23}) \cup I_3$ that has $(\{i\} \times A) \cup I_3$ as its group of size 8. (Note that such a design exists by lemma 8). Finally construct a $GD(4,1,\{2,5^*\};23)$ on $(I_4 \times A) \cup I_3$. This yields a $GD(4,1,\{2,5^*\};95)$ as required. □

LEMMA 24. $191 \in GD(4,1,\{2,5^*\})$, i.e. $31 \in V$.

PROOF. We shall construct a $GD(\{3,4\},1,2;132)$ where the triples form 59 Δ-factors.

(a) A 44 × 44 latin square with 5 increasing diagonals.
A transversal of a latin square is called an increasing diagonal if it is parallel to the main diagonal, and each entry is one more than the one immediately left-above it (here rows, columns and entries are thought of as elements of the cyclic group $Z_n$).
For instance 021 and 02413 are latin squares where all (3 resp.5) diagonals

| | |
|---|---|
| 021 | 02413 |
| 210 | 41302 |
| 102 | 30241 |
| | 24130 |
| | 13024 |

are increasing. For even orders such latin squares do not exist. However, 0231 has one increasing diagonal.
3102
1320
2013
Forming the direct product with an 11 × 11 LS with 11 increasing diagonals yields a 44 × 44 LS with 11 increasing diagonals. (The symbols here are (0,0),(0,1),(0,2),(0,3),(1,0),...,(10,3) in this sequence.)
Even more is true: 0231 and 0213 are mutually orthogonal, showing that there

| | |
|---|---|
| 0231 | 0213 |
| 3102 | 2031 |
| 1320 | 1302 |
| 2013 | 3120 |

is a RT (3,1;4) with 1 cyclic parallel class, and by taking the direct product with an 11 × 11 LS with 11 increasing diagonals (i.e. a cyclic RT(3,1;11)) we get a RT (3,1;44) with 11 cyclic parallel classes.

(b) The construction.
Let $X = I_3 \times Z_{44}$. Take a resolvable transversal design RT (3,1;44) with 5 cyclic parallel clases on X. Use 39 of its 44 parallel classes as they are, leaving 5 cyclic sets $\{(0,a_i),(1,b_i),(2,c_i)\}$ mod 44 (i = 1,2,3,4,5) whose triples will be distributed differently over the remaining 20 Δ-factors we still have to form. Next cover each $\{i\} \times Z_{44}$ ($i \in I_3$) as follows:

(α) take the matching {0,22} (mod 44)/2.

(β) take the quadruples {0,4,20,25} (mod 44).

(γ) take the triples {0,12,27},{0,8,10},{0,3,9},{0,7,18},{0,1,14}, all mod 44.

Now all we have to do is to form the remaining 20 Δ-facotrs. Each cyclic set of triples within $\{i\} \times Z_{44}$ $(i \in I_3)$ together with a cyclic set from the RT (3,1;44) will yield 4 Δ-factors. As follows:

If we have the 'horizontal' triple {0,p,q} and the 'vertical' one $\{(0,u_0),(1,u_1),(2,u_2)\}$ then form one Δ-factor by taking on $\{i\} \times Z_{44}$: $\{0,p,q\} + u_i + \lambda_j$ $(0 \le j \le 10)$ where λ is chosen such that the 33 numbers $0 + \lambda j$, $p + \lambda j$, $q + \lambda j$ are all different (and in particular $(\lambda,11) = 1$).

This leaves 11 points on each $\{i\} \times Z_{44}$, one in each congruence class mod 11.

Since they are shifted the right amount $u_i$ they form 11 blocks from $\{(0,u_0),(1,u_1),(2,u_2)\}$, thus completing the first Δ-factor.

Shifting all blocks by 11, 22, or 33 gives three more.

Remains to show that λ can be chosen suitably.

For {0,12,27} choose λ = 1,

for {0,8,10} choose λ = 3,

and for the other three triples choose λ = 4. □

By lemma's 1,11,16,17 and 18 we now know that $2v \in V$ iff $v \neq 1$.

Consider the case of odd m, and distinguish cases according to the residue class of m (mod 8).

a) $m \equiv 1 \pmod 8$, $m \neq 1$.

| m | proof of $m \in V$ |
|---|---|
| 9 | lemma 21 |
| 17 | lemma 12 |
| 25 | lemma 15 |
| 33 | lemma 13 |
| 41 | lemma 12 |
| ≥49 | lemma 4'. |

b) $m \equiv 3 \pmod 8$

| m | proof of $m \in V$ |
|---|---|
| 3 | lemma 19 |
| 11 | lemma 12 |
| ≥19 | lemma 4'. |

c) $m \equiv 5 \pmod 8$

| m | proof of $m \in V$ |
|---|---|
| 5 | lemma 12 |
| 13 | lemma 22 |
| 21 | lemma 14 |
| ≥29 | lemma 4'. |

d) $m \equiv 7 \pmod 8$

| m | proof of $m \in V$ |
|---|---|
| 7 | lemma 20 |
| 15 | lemma 23 |
| 23 | lemma 12 |
| 31 | lemma 24 |
| ≥39 | lemma 4'. |

This completes the proof of theorem 4.

REFERENCES

1. BAKKER, R.D., & R.M. WILSON, *Nearly Kirkman Triple systems*, to appear in Utilitas Math.

2. BERMOND, J.-C., & J. NOVÁK, *personal communication.*

3. BOSE, R.C., E.T. PARKER & S.S. SHRIKHANDE, *Further results on the construction of Mutually Orthogonal Latin Squares and the Falsity of Euler's Conjecture*, Can. J. of Math. 12 (1960) 189-203.

4. BROUWER, A.E., A(17,6,4) = 20 *or the nonexistence of the scarce design* SD(4,1;17,21), Math. Centre report ZW62, dec 1975.

5. BROUWER, A.E., H. HANANI, & A. SCHRIJVER, *Group-divisible designs with block size 4*, submitted to Discr. Math.

6. HANANI, H., *The existence and construction of balanced incomplete block-designs*, Ann. Math. Stat. 32 (1961) 361-386.

7. HANANI, H., *Balanced incomplete block designs and related designs,* Discr. Math. 11 (1975) 255-369.

8. HANANI, H., D.K. RAY-CHAUDHURI, & R.M. WILSON, *On resolvable designs,* Discr. Math. 3 (1972) 343-357.

9. HEDAYAT, A., *A set of Three Mutually Orthogonal Latin Squares of Order* 15, Technometrics 13 (1971) 696-698.

10. JOHNSON, S.M., *A new upper bound for error correcting codes,* IRE Trans. Information Theory, IT-8 (1962) 203-207.

11. KOTZIG, A., & A. ROSA, *Nearly Kirkman systems,* in: *Proc. 5th S-E Conf. on Combinatorics, Graph Theory and Computing,* Boca Raton, Fla., feb 1974, pp. 607-614.

12. LINT, J.H. VAN, *Combinatorical Theory seminar,* Springer Verley lecture notes no. 382, 1974 (especially pages 111-119).

13. MILLS, W.H., *Three mutually orthogonal Latin Squares,* J. Combinatorial Theory (A) 13 (1972) 79-82.

14. ——————, *On the covering of pairs by quadruples I, II,* J. Combinatorial Theory (A) 13 (1972) 55-78 and 15 (1973) 138-166.

15. RAY-CHAUDHURI, D.K., & R.M. WILSON, *Solution of Kirkman's schoolgirl problem,* in: *Proceedings of symposia in pure mathematics vol. 19: Combinatorics,* T.S. Motzkin (ed.), Amer. Math. Soc., Providence, R.I., 1971, pp. 187-204.

16. SCHÖNHEIM, J., *On maximal systems of K-tuples,* Studia Scientiarum Mathematicarum Hungarica 1 (1966) 363-368.

17. SLOANE, N.J.A., *personal communication.*

18. WILSON, R.M., *The construction of group divisible designs and partial planes having the maximum number of lines of a given size,* Proc. Chapel Hill 1970, pp. 488-497.

19. ——————, *A few more squares,* in: *Proceedings 5th S-E Conf. on Combinatorics, Graph Theory and Computing,* Boca Raton, Fla., feb 1974, pp. 675-680.

20. ——————, *Construction and uses of pairwise balanced designs*, in: *Combinatorics, MC tract 55, M. Hall, Jr. & J.H. van Lint (eds.)*, Mathematical Centre, Amsterdam and Reidel, Dordrecht, 1974, pp. 18-41.

21. ——————, *personal communication.*